Paul Samuelson's Legacy *

by

Avinash Dixit

Princeton University

E-mail: dixitak@princeton.edu

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I thank the editors for inviting me to write this article; it gave me a wonderful opportunity for re-reading much of Samuelson's work, and for reliving personal memories of him as a teacher and scholar. I am extremely grateful to Robert Solow for invaluable conversations, advice, and comments on the first draft, to Janice Murray, Samuelson's longtime assistant in the MIT economics department, for factual and editorial advice, and Peter Diamond and James Poterba for useful comments on the first draft.

Alexander Pope's epitaph for Isaac Newton was:

Nature and Nature's laws lay hid in night:
God said, "Let Newton be!" and all was light. [i]

The same could be said of Paul Samuelson. Many principles of economics were hidden in obscure verbiage of previous generations; he reformulated and extended them with crystal clarity in the language of mathematics. The citation for his Nobel Prize read: "for the scientific work through which he has developed static and dynamic economic theory and actively contributed to raising the level of analysis in economic science". [ii] He left his mark on most fields of economics. He launched new fields, and revolutionized stagnant fields. And he spanned this amazing breadth without the least sacrifice of depth in any of his endeavors.

If the depth of Samuelson's research resembles Newton's, the volume of his output rivals Bach, Picasso, or Trollope. In every one of the eight decades since the 1930s, he made fundamental contributions that enlightened, corrected, and challenged the rest of us. His collected works in seven massive volumes comprise 597 items. He firmly believed that it was a scientist's duty to communicate his knowledge to the profession; that it was "a sin not to publish" (Samuelson 1982 [1986], 98). [iii]

He molded several generations of graduate students at MIT and researchers throughout the profession. His introductory textbook guided the thinking of millions throughout the world; it was instrumental in spreading the Keyensian revolution; it was a model that all subsequent textbooks followed. His advice to presidents and his popular writings helped shape policy. More than anyone else in the latter half of the twentieth century, Samuelson changed the way economists think and write.

His deeds have not gone unsung. The Festshrift volume following the conference to celebrate his 65th birthday (Brown and Solow 1983) took a very unusual form; ten authors, themselves stellar, reviewed his contributions to, and his influence on, ten different major fields of economics. A second Festschrift resulting from the celebration of his 90th birthday (Szenberg, Ramrattan and Gotesman 2006) also had some chapters of this kind. These chapters provide excellent and thorough statements of Samuelson's research and its impact. That makes my job simultaneously very easy and very difficult. I need not give much detailed exposition or explanation of his work, but even brief sketches may be too repetitive. My solution (a highly constrained optimum?) is to be idiosyncratic and selective, emphasizing my favorites. I beg the readers' pardon in advance for any neglect of, or only passing mentions of, their favorites.

1. BRIEF BIOGRAPHICAL SKETCH [iv]

Paul Anthony Samuelson was born on May 15, 1915 in Gary, Indiana, and died on December 13, 2009 in Belmont, Massachusetts. [v] He enrolled at the University of Chicago in the 1931-32 academic year. His first economics course was from Aaron Director. He took several graduate courses, where his classmates included Milton Friedman. He got his bachelor's degree at the early age of 20, and entered the graduate program at Harvard. The following year he was chosen as a Junior Fellow of the elite Society of Fellows -- the first economist to earn this honor. The freedom to pursue his own research in that position yielded an amazing stream of pathbreaking papers on the theory of consumer choice, intertemporal structure of prices and interest rates, and many other areas, culminating in his prize-winning dissertation in 1941, and the landmark book *Foundations of Economic Analysis* in 1947. He described this book as "thrice-blessed" (Samuelson 1986 a [1986], 802) -- it won him Harvard's David A. Wells dissertation prize and the American Economic Association's first Clark Medal, and was a major factor in his being awarded the Nobel Prize.

He started as an Instructor at Harvard in Fall1940, but within a month left for an assistant professorship at MIT. The reasons, according to Breit and Ransom (1982, 110), included his "youth, brash personality, and Jewish background." Harvard was certainly anti-semitic in those days; it had an upper-bound quota for Jewish students, and very few Jewish faculty members. Many in Harvard's economics department, especially the chairman Harold Burbank, were also anti-mathematics. Samuelson (1983 a, 11) himself makes it seem a very simple decision: "I left Harvard in 1940 for the same reason that James Tobin left it in 1950: I got a better offer." He also quotes advice from his revered teacher E. B. Wilson: "Until you leave home, he said in effect, you are a boy." However, he also hints that renewal of his temporary Harvard instructorship was unlikely: "I never received any letter, oral promise, or clairvoyant message even hinting that a permanent appointment awaited me" (1983 a, 12).

Whatever the reasons, the move to MIT worked out very well for Samuelson and MIT alike. He stayed there for the rest of his long career, becoming Institute Professor in 1966, which, with its freedom to teach what and when he wanted, he described as a "perpetual Junior Fellowship" (Samuelson 1986 a [1986], 803).

Early in his MIT career, at the urging of department head Ralph Freeman to write an introductory textbook that MIT's engineering undergraduates would like, he worked for three years to produce *Economics,* which was published in 1948 (Samuelson 1948 a). It became a great success worldwide, went through numerous editions and translations, and was a major influence in spreading the Keynesian revolution to the profession and the public.

Samuelson made his huge impact on economics in more ways than through his writings. He was *primus inter pares* in MIT's economics department for decades. He set a style and example of dedication and collegiality that numerous students and former colleagues have taken with them to other schools. His travels, and voluminous correspondence with economists in many countries, enabled the profession worldwide to benefit from his wisdom and experience. He never sought any direct policymaking role, and turned down invitations to serve in such positions. But in a less formal capacity he advised presidents Kennedy and Johnson and many other top policy-makers. He was a consultant to the United States Treasury, the Bureau of the Budget and the President's Council of Economic Advisers. He was an active participant in policy debates, perhaps most importantly through his columns in *Newsweek.*

I do not have the space to list the honors Samuelson received -- his honorary degrees, his memberships in all the major academies, the presidencies of professional societies, and so on -- and will limit myself to mentioning just three: the inaugural John Bates Clark Medal in 1947, the Nobel Prize in 1970, and the National Medal of Science, considered America's top science honor, in 1996.

2. CONSTRAINED OPTIMIZATION AND CHOICE

Samuelson's *Foundations of Economic Analysis* (Samuelson 1947, henceforth just *Foundations*) treats many topics in economic theory, but its most pervasive themes are the mathematics of constrained optimization and its implications for refutable hypotheses concerning behavior. Samuelson himself attached primary importance to these themes. His Nobel Prize lecture (Samuelson 1970 [1972]) begins: "The very name of my subject, economics, suggests economizing or maximizing" and continues, with many illustrations and digressions, "a maximum formulation permits one to make clear-cut inferences about a complicated system involving a large number of variables." He regarded this as the essential requirement of a scientific theory: it should have implications -- Samuelson calls them "meaningful theorems" -- that then become testable hypotheses.

Samuelson's interest in consumer theory began early -- perhaps the most important article is 1938 a [1966] -- and continued throughout his life. It occupies a central place in *Foundations*; three full chapters and many sections of other chapters are devoted to it. Using the framework of maximizing an ordinal utility function subject to a budget constraint, Samuelson derives many meaningful theorems, such as

(1) the own substitution effect is negative,

(2) cross-substitution effects are symmetric,

(3) the envelope theorem, and

(4) the Le Chatelier Principle.

Of these, the last two have very broad applicability, and bear a formal restatement, especially because Houthakker's (1983) essay on consumption theory in the Brown and Solow (eds) (1983) Festschrift emphasizes different aspects.

I will consider the simple problem of choosing $(x_1, x_2, \cdots)$ to maximize $v = f(x_1, x_2, \cdots, \theta)$, where θ is an exogenous parameter. I will assume that the function f is sufficiently differentiable, and that the optimal choices are unique functions of the parameter, $(x_1(\theta), x_2(\theta), \cdots)$. The principle is valid far more generally, but the simpler setting will suffice to convey the basic ideas.

Figure 1 shows the resulting maximized value $f(x_1(\theta), x_2(\theta), \cdots, \theta)$ as a function of θ. Take any particular value θ_0, and consider the function $f(x_1(\theta_0), x_2(\theta_0), \cdots, \theta)$, where the choice variables are held fixed at their optimal values corresponding to θ_0, and only the parameter's direct effect on the value of the function is allowed. Call this the fixed-x value function, and the other the optimum value function. At θ_0 the two functions coincide, but everywhere else the optimum value function must be at least as high as the fixed-x value function. This follows from the concept of maximization itself. In most nontrivial applications the optimum value function will be strictly above the fixed-x value function when $\theta \neq \theta_0$; this is the situation shown in the figure. We immediately see that at θ_0 the two curves must be mutually tangential (have the same slope), and the optimum value curve must be more convex (less concave) than the other.

If we do this for different values of θ_0, the optimum value function will be the upper envelope of all the fixed-x value functions; this is the envelope theorem. Its application to consumer theory comes from considering the problem of minimizing the expenditure needed to achieve a given utility level:

$$\text{minimize } p_1 x_1 + p_2 x_2 + \cdots \quad \text{subject to} \quad U(x_1, x_2, \cdots) \geq u_0,$$

in obvious notation. Call the resulting minimum value function the dual expenditure function $E(p_1, p_2, \cdots, u_0)$. Letting p_1 play the role of θ, and taking any one particular vector of prices $(\bar{p}_1, \bar{p}_2, \cdots, u_0)$, the fixed-$x$ value function becomes

$$p_1\, x_1(\bar{p}_1, \bar{p}_2, \cdots, u_0) + p_2\, x_2(\bar{p}_1, \bar{p}_2, \cdots, u_0) + \cdots .$$

The expenditure function allows reoptimization of the x's as prices change; therefore it lies below (this is a minimization problem) the fixed-x value function. The tangency condition becomes

$$x_1(\bar{p}_1, \bar{p}_2, \cdots, u_0) = \partial E / \partial p_1 \text{ evaluated at } (\bar{p}_1, \bar{p}_2, \cdots, u_0),$$

which is Shepherd's Lemma giving the compensated demand functions. This forms the basis of the theory of prices indexes and chaining that governs the practice of cost-of-living adjustment in many wage contracts and pension schemes.

Next, the fixed-x value function is linear in p_1, and the expenditure function lies below it, so the latter must be concave. Therefore

$$\partial x_1(\bar{p}_1, \bar{p}_2, \cdots, u_0) / \partial p_1 = \partial^2 E / \partial p_1^{\,2} \;\leq\; 0,$$

which signs the own substitution effect. And

$$\frac{\partial x_1(\bar{p}_1,\bar{p}_2,\cdots,u_0)}{\partial p_2} = \frac{\partial}{\partial p_2}\left(\frac{\partial E}{\partial p_1}\right) = \frac{\partial}{\partial p_1}\left(\frac{\partial E}{\partial p_2}\right) = \frac{\partial x_2(\bar{p}_1,\bar{p}_2,\cdots,u_0)}{\partial p_1},$$

which proves symmetry of cross-substitution effects.

Now go back to the general theory of Figure 1, and consider an intermediate value function where, as θ changes, x_1 is held fixed at its optimum value corresponding to θ_0, but the other x's are allowed to vary to their new optimum levels. The resulting value function must lie between the fully optimal one and the fixed-x one. Therefore its curvature must be intermediate between them. Applying this to the consumer's expenditure-minimization problem (with reversals of signs of curvatures as appropriate for minimization), and letting p_2 take the role of θ, we have

$$\left.\frac{\partial x_2(\bar{p}_1,\bar{p}_2,\cdots,u_0)}{\partial p_2}\right|_{x_1 \text{ free}} = \left.\frac{\partial^2 E}{\partial {p_2}^2}\right|_{x_1 \text{ free}} \leq \left.\frac{\partial^2 E}{\partial {p_2}^2}\right|_{x_1 \text{ fixed}} = \left.\frac{\partial x_2(\bar{p}_1,\bar{p}_2,\cdots,u_0)}{\partial p_2}\right|_{x_1 \text{ fixed}} \leq 0.$$

In other words, the own substitution effect for any commodity has smaller magnitude (and negative sign) when the quantity of some other commodity is fixed than when it can also be varied to minimize expenditure. Roughly speaking, a quantity constraint on the consumption of any one commodity reduces (in numerical magnitude) the price elasticity of demand for any other commodities. This is true regardless of whether the two commodities are substitutes or complements.

This is an example of the Le Chatelier Principle in economics. While the other three theorems developed above existed in some form before Samuelson in the work of Slutsky, Hicks, and Allen, he should get the sole credit for this principle. He named it by analogy with thermodynamics, where the effect of a change in the volume of a body of gas on its pressure is less when the temperature is held constant than when entropy is held constant and the temperature is allowed to adjust and equilibrate with the surroundings.

I am not saying that *Foundations* derives these theorems so slickly; my exposition has benefitted from much subsequent development of duality theory by McFadden and others. But many elements of this approach are to be found in *Foundations* and in Samuelson's later work, particularly his 1953 [1966] paper discussed below in the section on international trade.

The thermodynamics analogy prompts two remarks. First, Samuelson's "revered teacher of mathematical economics and statistics" during his Harvard years, E. B.

Wilson, had been the "favorite student" of Willard Gibbs, the theoretical physicist, physical chemist and mathematician (Samuelson 1983, 11). Gibbs was renowned for developing the theory of thermodynamics as an equilibrium system with many interrelated variables, and characterizing how the whole equilibrium changes when some exogenous conditions (parameters) change. This perspective of equilibrium and its comparative statics pervades much of Samuelson's thinking.

Second, Samuelson's scientific philosophy, with its insistence on using theory to derive refutable implications or meaningful theorems, shows some influence of logical positivism. But it goes beyond that into the methodological position of "operationalism" developed by the 1946 physics Nobel laureate Percy Bridgman. For Bridgman, entities can only be defined through the operations by which they are measured. For example, "temperature" is defined by the result of measurement using a thermometer. This perspective made Samuelson a dedicated ordinalist. He could see no way in which measurement on a consumer's behavior could lead to a measurement of utility; therefore cardinal utility was simply undefinable. Throughout his life he persisted in reexamining situations that seemed to call for cardinal utility, such as choices under risk, to recast them using measurable alternatives like "risk-corrected certainty equivalents" (Samuelson 1997 a b [2011]).

Samuelson's theory of revealed preference -- how a consumer's whole preference relation, or indifference map, can be inferred to any desired level of accuracy by observing his choices for sufficiently many configurations of prices and incomes -- can be thought of as the mirror-image of his insistence on using utility maximization subject to the budget constraint to derive meaningful theorems on choice. He went a long way on this path, but it remained for Houthakker (1950) to complete the journey. He developed the strong axiom of revealed preference, namely that for a chain of any length n of price vectors $p^1, p^2, \cdots p^n$ and the corresponding quantity vectors $x^1, x^2, \cdots x^n$ chosen by the consumer,

$$\text{if } p^1 \bullet x^1 \geq p^1 \bullet x^2, \cdots p^{n-1} \bullet x^{n-1} \geq p^{n-1} \bullet x^n, \text{ then } p^n \bullet x^n < p^n \bullet x^1,$$

where the dot denotes the inner product. (In words, if at prices 1 the consumer could have afforded quantities 2 but by his choice revealed quantities 1 to be preferable, and so on, then at prices n he could not have afforded quantities 1.) He proved that it ensures recovery of a consistent preference relation from observed choices. Samuelson (1950, [1966]) in turn completed Houthakker's analysis in the continuous formulation, establishing the symmetry of cross-substitution effects as the "integrability" condition that yields an ordinal utility function from demand functions. For many years these ideas remained mostly at the conceptual level, but work by Varian and others, described in Varian (2006), has brought it to the empirical and applied realms.

Many of Samuelson's pathbreaking contributions to economic theory build on the foundations of constrained maximization. Therefore one wonders what he thought of psychological and behavioral economics, some of which denies maximization of any complete, transitive objective function. Unfortunately none of his writings give us a clue, so I must speculate. He did not regard maximization as the literal description of the process of individual choice; his Nobel lecture (Samuelson 1970 [1972], 4-5) carefully states the "as if" nature of the theory, drawing analogy with the least action principle in physics. He recognized that there are important parts of economics "that can in no useful sense be related to a maximum problem" (Samuelson 1970 [1972], 13). And he declared: "I am primarily a theorist. But my first and last allegiance is to the facts" (Samuelson 1983 b [1986], 791). Therefore I believe that he would have welcomed new findings on how people actually make choices. But he would have insisted on the same standards of rigor and testing as in any empirical work; he would have resisted the tendency to accept every new claim made by some behavioristas, based on results from a few experiments where small and highly selected samples of subjects make choices in an artificial and unfamiliar structured environment, often influenced by what they think the experimenter expects of them.

3. GENERAL EQUILIBRIUM AND WELFARE ECONOMICS

Samuelson's interest in general equilibrium was primarily in specific contexts such as international trade, finance, or life-cycle saving. He saw few operational implications in the abstract system or structure that stood behind these specific applications. He did, however, have full command of the structure, and its implications for efficiency and welfare properties. This is perhaps best seen in his masterly RAND memorandum (Samuelson 1949 a [1966]), which exposits linear and dynamic programming from an economist's perspective of constrained maximization, efficient quantity choices and their dual supporting prices. The chapter on welfare economics in *Foundations* also rests on solid foundation of general equilibrium thinking.

A minor but interesting example of his grasp of general equilibrium can be seen in his comment (Samuelson 1972 [1986]) on Waugh, who had argued that price instability benefits consumers, basically because consumer surplus (or the indirect utility function in a quasi-linear model) is a convex function of prices. Samuelson pointed out that prices are endogenous variables in general equilibrium, and feasible changes in exogenous conditions that created the price instability would always raise the average price by more than enough to wipe out the gains from the variance.

Stability of general equilibrium was one of Samuelson's major concerns, and what he called the "correspondence principle" one of his important contributions to it. Under the standard price adjustment process where the rate of increase of each price is proportional to the excess demand for that commodity, local stability of the price vector requires all eigenvalues of the matrix of derivatives of excess demand functions to have negative real parts. If excess demand functions result from consumer and producer maximization, the second order conditions pertain to definiteness of the same matrix (with usual complications arising from income effects in the case of the consumer); this has some implications for comparative statics properties of the system. The two properties of the matrix are related but not identical, and stability conditions yield some additional meaningful restrictions on comparative static derivatives. Samuelson explored this in great detail (Samuelson 1947, chapter IX). Hahn (1983) discusses the work on general equilibrium and stability, including some related capital theory.

Samuelson's work on welfare economics was also influenced by his ordinalism and by his deep understanding of general equilibrium in action. Chapter VIII of *Foundations* develops what is now called the Bergson-Samuelson formulation of optimality conditions in full generality, always emphasizing that the whole setup is perfectly consistent with ordinal utility: any monotonic transformations of individual utility functions can and should be undone by offsetting transformations in the social welfare function so as to leave the substantive interpersonal comparisons unchanged

(1947, 228) and "only ratios of marginal utilities for the same individual are involved" in the Pareto optimality conditions (1947, 237).

He also wrote on the links between the Bergson-Samuelson welfare function and social choice theory; see Arrow's masterly discussion (1983).

4. INTERNATIONAL TRADE

Samuelson made seminal contributions to almost all fields of economics, but international trade theory was always one of his favorite areas of research. He visited it in almost every decade of his career, and always found something new and thought-provoking to say.

He saw from the earliest days that trade theory and general equilibrium theory were very closely linked, and that the linkages flowed both ways. The opening sentence of one of his earliest publications (Samuelson 1938 b [1966]) is: "Historically the development of economic theory owes much to the theory of international trade." He used the Ricardian model of comparative advantage as the starting point of his discussion of linear programming (Samuelson 1949 a [1966]). And recently he said (Samuelson 1995, 17): "If you've got it, flaunt it. Well, we in trade theory do have a lot to display. ... Thus, the first general equilibrium was not by Léon Walras. Sixty-five years earlier it was by John Stuart Mill – and in connection with international equilibrium. ... Stan Ulam [vi] ... once challenged me, saying 'Paul, name me one proposition in the social sciences that is both true and non-trivial.' My reply was: 'Ricardo's theory of comparative advantage.' ... Trade theory leads the way."

Samuelson's papers on international trade exemplify his skill in building small models tailored to pose and answer specific questions about the functioning of a specific aspect of the economy. Most of them are mini general equilibrium systems, but in a concrete form, focusing on the essentials at issue and simplifying or ignoring many other aspects. This style -- labeled MIT, or toy modeling by admirers and detractors alike -- has proved highly successful in the hands of former students like Diamond, Stiglitz, and Krugman, and has spread widely throughout the profession. Much of my own research has also been influenced by it.

He started in the 1930s with definitive work on gains from trade. In the 1940s he established the 2-by-2 model, now called the Heckscher-Ohlin-Samuelson model, as the standard tool in international trade. The 1950s brought what is my personal favorite article, on the interaction between the prices of goods and factors in a general equilibrium with trade. It extended much of the earlier analysis beyond the 2-by-2 case, and gave us new tools, particularly duality and the revenue or GDP function. Samuelson continued the theme of "beyond 2-by-2" with the sector-specific factor model, now called the Ricardo-Viner-Samuelson model, in the late 1960s and the early 1970s, and the Dornbusch-Fischer-Samuelson model of trade with a continuum of commodities in the late 1970s; both models are now an essential part of the international trade economists' toolkit. Finally, in his eighties and in the new century, he engaged in policy controversies concerning globalization and outsourcing. Ronald Jones' (1983) excellent survey of Samuelson's impact on trade theory saves me the need to go into details; therefore I shall merely offer some reinterpretation and updating.

4.1 Gains from Trade

Samuelson's early articles on the gains from trade (1938 b and 1939 a [1966]), and a later one (1962 b [1966]), laid down the basic comparisons (free trade versus autarky, free trade versus restricted trade, Pareto efficiency versus social optimality) and techniques (revealed preference inequalities, consumption and utility possibility frontiers) that have guided thinking on this issue ever since. Most importantly, Samuelson showed that the utility possibility frontier with free trade lies outside the utility possibility frontier with autarky, provided the aggregate quantities of goods available in both situations can be distributed among the country's consumers by the government. Thus, given any allocation under autarky, a move to free trade accompanied by suitable individual-specific lump-sum transfers can achieve a Pareto superior outcome; whereas, given any allocation under free trade, no move to autarky accompanied by any redistribution can be Pareto improving. The transparency of the analysis -- it does not involve any arcane questions about the nature of costs -- and its generality -- it does not depend on there being only two goods, or on all goods being tradable, etc -- swept away much previous confused thinking on the subject. Subsequently, Grandmont and McFadden (1972) gave more rigorous proofs of Pareto superiority of trade with individualized lump-sum transfers, and Dixit and Norman (1986) clarified when and how Pareto superiority of trade can be achieved using only Ramsey-Diamond-Mirrlees commodity taxation.

Jumping ahead, Samuelson (2004 [2011]) used a Ricardian model and numerical examples to illustrate how a trading country's welfare can go down if another country raises its productivity in this country's export good. This is in fact a far more general result; the simple intuition is that the shift worsens this country's terms of trade. This is a useful cautionary piece, teaching us to guard against some exaggerated claims or suggestions that under globalization every day in every way everything gets better and better. However, it does not imply any support for protectionism: gains from trade are still positive, only smaller than they were before our exports faced stiffer competition.

4.2 Factor Endowments and Trade

Samuelson seems to have regarded differences of factor endowments among countries as a better explanation of trade than Ricardian productivity differences. In his first paper on factor price equalization (Samuelson 1948 b [1966]), he says: "instead of relying upon such crypto-explanations as 'Yankee ingenuity' to explain patterns of comparative advantage, Ohlin would attribute America’s comparative advantage in food production -- a land-intensive industry -- to the fact that each unit of American labor has relatively much land to work with."

The resulting two-good, two-factor model is now justly called the Heckscher-Ohlin-Samuelson model. His work produced two of the “four theorems of trade theory.” With Stolper (1941 [1966]), he showed how changes in the international prices of goods lead to magnified changes in the domestic prices of factors, and generated unambiguous predictions about the effects of tariffs on the real returns to factors. This came to be called the Stolper-Samuelson effect. In two papers (Samuelson 1948 b and 1949 b [1966]) he found that free trade in goods leads to complete equalization of the prices of factors, even though factors trade in separate country-specific markets, so long as the factor endowment proportions in the two countries are not too different. The question is whether the system of nonlinear equations equating the unit cost of each good being produced in a country to the world price of that good has a unique solution for the domestic input prices. This analysis of "global univalence," with more general mathematical analysis in an appendix to Samuelson (1953 [1966]), actually led to some new mathematics -- a global inverse function theorem. Hahn (1983, 44-48) gives details of the developments and Mas Colell (1979) a rigorous mathematical treatment.

Samuelson also makes a purely verbal argument that gets much more directly at the economics factor price equalization, and illustrates how Samuelson's technical power and his intuition ran hand-in-hand. This is the wonderful "angel and recording geographer" device: "Let us suppose that in the beginning all factors were perfectly mobile, and nationalism had not yet reared its ugly head. ... [T]here would be one world price of food and clothing, one real wage, one real rent, and the world's land and labour would be divided between food and clothing production in a determinate way, with uniform proportions of labour to land being used everywhere in clothing production, and a smaller—but uniform—proportion of labour to land being used in production of food. Now suppose an angel came down from heaven and notified some fraction of all the labour and land units producing clothing that they were to be called Americans, the rest to be called Europeans. ... Obviously, just giving people and areas national labels does not alter anything; it does not change commodity or factor prices or production patterns. ... [W]hat will be the result? Two countries with quite different factor proportions, but with identical real wages and rents and identical modes of commodity production (but with different relative importance of food and clothing industries). ... Both countries must have factor proportions intermediate between the proportions in the two industries. The angel can create a country with proportions not intermediate between the factor intensities of food and clothing. But he cannot do so by following the above-described procedure, which was calculated to leave prices and production unchanged." (Samuelson 1949 b [1966], 882-883).

Expressed thus, the intuition is very similar to that of spanning in financial markets (Ekern and Wilson 1974). The efficient outcome of complete markets in Arrow-Debreu contingent claims for all states of the world can be replicated by linear combinations of more familiar securities like equities and bonds, if the vectors of payoffs of the latter securities span the whole space of Arrow-Debreu contingent claims. Similarly, when countries differ in their factor endowments, full production efficiency could be achieved by letting them trade factor services directly. But suppose they can only trade prepackaged bundles of these factors, namely those embodied in units of each of the goods. This suffices if the vectors of these bundles of factor services span the factor space. Of course this is an equilibrium concept. In finance one must find the real choices of firms to know the patterns of profits in the available securities and see if they span the full space; in trade one must solve for the factor proportions in the hypothetical equilibrium of an integrated world with international factor mobility and see if these factor bundles suffice for the purpose. There is an important difference between finance and trade: finance theory usually allows short sales of securities, while production quantities in trade must be inherently non-negative in each country. Therefore we must incorporate this restriction and allow only non-negative spanning; in the 2-by-2 model this requires that the factor proportions in the two countries should not be too different. But the analogy captures well the economic idea that trade in goods is an indirect way of trading factor services, and generates more useful intuition than the mathematics of univalence. This approach to factor price equalization was developed for the competitive factor endowment models by Dixit and Norman (1980, 110-125, 289-291), and was used in many other contexts including foreign direct investment by Helpman and Krugman (1985) and others.

The Stolper-Samuelson and factor price equalization papers did not actually produce the Heckscher-Ohlin theorem, namely the prediction that the pattern of trade will correspond to relative factor abundance, although the idea was implicit there. As Jones (1983, 89) says, "it was left to the next generation to explore this 2×2 model in more detail for the effect of differences in factor endowments and growth in endowments on trade and production patterns." That, plus the Rybczynski theorem which arose independently, completed the famous four theorems. Jones' own article (1965) is my favorite exposition of the complete story.

All this and much more came together in Samuelson (1953 [1966]). With any number of goods and factors, he established a duality between prices and quantities (more precisely, a reciprocity relationship linking the Stolper-Samuelson and Rybczynski effects), and studied the univalence question with any number of goods and factors. In the process, he developed tools, most notably the revenue or GDP function, that have found numerous uses in trade theory. This paper on its own clinches the link between trade theory and general equilibrium theory.

4.3 The Transfer Problem

If one country makes a transfer of income to another (in the form of a gift or reparations), will it suffer a secondary loss because its terms of trade deteriorate? This was the orthodox presumption in the debate about German reparation payments after World War I, and was supported by Keynes and others, although it was contested by Ohlin. The arguments were based on supposed low price elasticities of demand for German exports. Samuelson (1952 a and 1954 a [1966], 1971 b [1972]) clarified and simplified the issue. In a simple two-country (Germany and the rest of the world) and two-good (German exports and imports), at the initial prices, the impact effect of the transfer is to increase income in the rest of the world and lower it in Germany by equal amounts. This will lower worldwide demand for German exports if the rest of the world's marginal propensity to spend its income on the German export good is less than Germany's own marginal propensity to spend on its own export good. If this is the case, the relative price of German exports will fall -- its terms of trade will deteriorate -- in order to restore equilibrium, so long as the equilibrium is stable. So the relevant comparison is between marginal propensities to spend; price elasticities are irrelevant. At this level of generality there is no reason to suppose which marginal propensity is larger, so the effect is ambiguous. Samuelson then examines departures from the basic model -- transport costs or trade barriers, non-traded goods or leisure creating substitution effects in supply as well as demand, and so on -- under which the orthodox presumption can be rescued.

4.4 Real Exchange Rates

Official exchange rates between currencies usually fail to reflect relative prices between the countries in question. Balassa (1964) and Samuelson (1964 a) gave an explanation of this departure from purchasing power parity (PPP). They pointed out that technological progress in richer or faster-growing countries is usually stronger in their traded good sectors. This draws resources away from the non-traded goods sectors, while the higher incomes increase the demand for non-traded goods. For both reasons, the relative prices of non-traded goods rise. The real exchange rate, which captures the ratio of domestic and international price indexes, therefore exceeds the nominal exchange rate for such countries. This idea underlies the whole enterprise of PPP-adjusted income comparisons. [vii]

5. CAPITAL AND GROWTH THEORY

Once again Samuelson (1937 a, 1937 b, 1939 b [1966] was one of the first to provide rigorous statement of the foundations of this field, namely the theory of intertemporal consumption and production decisions (including a sketch of a life-cycle model of saving!), prices and interest rates.[viii] His superior knowledge of mathematics, in this case integral calculus, enabled him to simplify the exposition and clear away much previous confusion. The same applies to his neat little note on depreciation (1964 b [1972]).

The quarter-century from about 1950 to about 1975 was the peak period for "neo" capital and growth theories -- neoclassical, neo-Marxian and other -- and associated controversies. [ix] Samuelson was at the forefront of all these developments. In my view his main substantive contributions were to the development of non-substitution theorems and turnpike theorems. I can leave out the details because his long-standing friend, colleague, coauthor and office-neighbor Solow has given us an insider's account (1983). Here is a brief sketch of the ideas and issues.

Non-substitution theorems tell us when equilibrium relative prices are independent of demand. The Ricardian trade model is the simplest case: two goods are produced by labor alone under constant returns to scale, so the production possibility frontier is a straight line. Its technologically determined slope fixes the relative price of the two goods; the input coefficients then fix the price of labor services relative to the goods. The idea can be generalized greatly; there can be any number of goods, use of some produced goods as inputs in production of other goods, and discrete or continuous choice of techniques in the production of any or all goods. Similar theorems for steady states of economies where production takes time and there can be circulating or fixed capital are also available. The key assumptions are constant returns to scale, no joint production, and only one non-produced input (such as labor).

Samuelson (1951 [1966]) stated one of the earliest of these theorems, and gave a very neat proof of it using his beloved methods of constrained optimization. Let 0 denote labor and $1,2,\cdots n$ the produced commodities. Let X_{ji} denote the input of j in production of i, and F^i the production functions, assumed to be concave and homogeneous of degree 1. Let X_0 be the total amount of labor available. Then an efficient production plan maximizes the net output of any one of the commodities, say 1,

$$C_1 = F^1(X_{01}, X_{11}, X_{21}, \cdots X_{n1}) - X_{11} - X_{12} - \cdots - X_{1n}$$

subject to producing given amounts $C_2, \cdots C_n$ of each of the other commodities:

$$F^i(X_{0i}, X_{1i}, X_{2i}, \cdots X_{ni}) - X_{i1} - X_{i2} - \cdots - X_{in} \geq C_i \quad \text{for } i = 2, \cdots n$$

and

$$X_0 \geq X_{01} + X_{02} + \cdots + X_{0n}\,.$$

Form the Lagrangian

$$\Lambda = \sum_{i=1}^{n} \lambda_i \left[F^i(X_{0i}, X_{1i}, X_{2i}, \cdots X_{ni}) - X_{i1} - X_{i2} - \cdots - X_{in} \right] - \lambda_0 \left[X_{01} + X_{02} + \cdots + X_{0n} \right],$$

where for symmetry of notation we take $\lambda_1 \equiv 1$.Then the first-order conditions, necessary and sufficient because of concavity, are

$$\lambda_i \frac{\partial F^i}{\partial X_{ji}} = \lambda_j \quad \text{for } i = 1,2,\cdots n,\ j = 0,1,2,\cdots n\,.$$

Using this with $i = 1$, we have $\partial F^1 \big/ \partial X_{j1} = \lambda_j$ for all $j = 0,1,2,\cdots n$, which enables us to substitute out for the λs:

$$\frac{\partial F^1}{\partial X_{11}} = 1, \quad \text{and} \quad \frac{\partial F^1}{\partial X_{i1}} \frac{\partial F^i}{\partial X_{ji}} = \frac{\partial F^1}{\partial X_{j1}} \quad \text{for} \quad i = 2,\cdots n, \quad j = 0,1,2,\cdots n\,.$$

This is a system of $1 + (n-1)(n+1) = n^2$ equations. Only the marginal product functions appear, and because of constant returns to scale they are homogeneous of degree zero in the inputs. Therefore they can be expressed as functions of input ratios alone, of which there are precisely n^2 independent ones, for example X_{ji} / X_{0i} for $i, j = 1,2,\cdots n$. With some additional assumptions to ensure the independence of the equations etc., the system will have a unique solution. The system does not involve the quantities C_i at all;

therefore the solution must be independent of the location the point $(C_1, C_2, \cdots C_n)$ on the frontier of efficient net outputs. In economic terms, the best technique for producing each good is independent of the output mix.

This solution also fixes the input coefficients

$$a_{ji} = X_{ji} / F^i(X_{0i}, X_{1i}, \cdots X_{ni}) \text{ for } i = 1, 2, \cdots n,\ j = 0, 1, 2, \cdots n .$$

Abbreviating the gross outputs by $G_i = F^i(X_{0i}, X_{1i}, \cdots X_{ni})$, the net outputs become

$$C_i = G_i - \sum_{j=1}^{n} a_{ji}\, G_j , \quad \text{or} \quad (\mathbf{I} - \mathbf{A})\, \mathbf{G} = \mathbf{C}$$

in obvious vector-matrix notation. Similarly the labor constraint becomes

$$X_0 = \sum_{j=1}^{n} a_{oj}\, G_j , \quad \text{or} \quad \mathbf{a_0}'\, (\mathbf{I} - \mathbf{A})^{-1}\, \mathbf{C} = X_0 .$$

This is just the familiar Leontief system, as if the sole efficient technique for producing each good were the only one available (hence the "non-substitution"). Therefore the production possibility frontier is a hyperplane, and the coefficients multiplying the net output quantities in the equation of the hyperplane are the direct and indirect labor requirements of all the commodities. Parallels to Ricardian or Marxian values are there for the taking.

Fully rigorous treatments require much dotting of i's and crossing of t's: ensuring the existence of a unique solution, allowing non-differentiable production functions and discrete sets of techniques, worrying about corner solutions with zero net outputs of some commodities, and so on. Time lags in production bring more complex issues of capital theory: rental and purchase prices, interest rates etc. All this was done later by Samuelson himself and others; Mirrlees (1969) is probably the best general treatment. But the simple model and derivation sketched above already convey much of the basic intuition behind the theorem.

In several papers, sole-authored and coauthored with Solow, and most importantly with coauthors Dorfman and Solow in *Linear Programming and Economic Analysis* (1958), Samuelson investigated the behavior of efficient growth paths in many-commodity models of production where the commodities can have multiple roles as outputs, intermediate inputs, and durable capital goods. Chapter 12 of the book has a beautifully simple and clear account of the intertemporal efficiency condition (at any time and between any two goods that are both produced and used in production, the marginal rate of output transformation should equal the marginal rate of input substitution). It also gives an intuitive explanation and sketch of a proof that efficient paths over long time-spans have a "turnpike property." In these models, there is a path of maximal proportional or balanced growth (the von Neumann ray). "Take *any* initial capital structure and *any* desired terminal structure. Then if the programming period is long enough ... [t]he system first invests so as to alter its capital structure *toward the special von Neuman proportions*. When it has come close to these proportions, it spends most of the programming period performing steady growth at the maximal rate (or more precisely, something close to maximal steady growth). The system *expands along or close to the von Neumann ray* until the end of the programming period approaches. Then it bends away from [this ray] and invests in such a way as to *alter the capital structure to the desired terminal proportions*. ... It is exactly like a turnpike paralleled by a network of minor roads. ... [I]f origin and destination are far enough, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end." (Dorfman, Samuelson and Solow 1958, 330-331, emphasis in the original) On p.334 they relate this to the saddle-point property of phase diagrams of growth paths. These ideas have become pervasive in growth and capital theories, and in related macroeconomic models in fields like public finance and international economics.

Against these impressive achievements must be set the long controversy about reswitching which, in my judgment, was a waste of much first-rate brain-power on both sides of the debate. Briefly stated, the issue is this. Consider an infinite-horizon economy whose technology is defined by a feasible set T of sequences of net output vectors $\mathbf{x} = (\mathbf{x}_t \mid t = \cdots -1,0,1,2,\cdots)$. Efficient sequences $\mathbf{x}$ are defined in the usual way. Interest focuses on technologies that permit efficient sequences with proportional growth, that is, for some g, $\mathbf{x}_t = (1+g)^t\ \mathbf{x}_0$ for all t.

Under some technical conditions, for each efficient $\mathbf{x}$ there exists a convex cone of supporting sequences of price vectors, that is,

$$\mathbf{P(x)} = \left\{\mathbf{p} = (\mathbf{p}_t \mid t = \cdots -1,0,1,2,\cdots) \,\middle|\, \mathbf{x} \text{ maximizes } \mathbf{p.x} \text{ over T}\right\}.$$

An efficient sequence of output vectors with proportional growth can be supported by at least one price sequence showing proportional decline, that is, there exists r such that

$\mathbf{p}_t = (1+r)^{-t}\ \mathbf{p}_0$ for all t. These are present value prices, so r is an own rate of interest, same for all commodities, and can be called "the" rate of interest.

Such a supporting price sequence need not be unique: the same proportional-growth quantity sequence may be supported by two proportional-decline price sequences with different interest rates, say r_1 and r_2. But a proportional-decline price sequence with an intermediate interest rate r_3 (with $r_1 < r_3 < r_2$) may support not $\mathbf{x}$, but a different proportional-quantity sequence say $\mathbf{x}'$. Thus the same sequence $\mathbf{x}$ is optimal at low and high interest rates r_1 and r_2, but a different sequence $\mathbf{x}'$ is optimal at the intermediate interest rate r_3. This phenomenon of the "reswitching" of techniques thus destroyed any hope of using interest rates as monotonic ordinal measures of properties like the capital intensity of techniques of production, a hope that neoclassical economists including Samuelson (1962 c [1966]) once held.

The supporting prices of a given efficient sequence form a convex cone, so a supporting price sequence that is a convex combination of two other supporting price sequences, say $\mathbf{p}_t^3 = \lambda\, \mathbf{p}_t^1 + (1-\lambda)\, \mathbf{p}_t^2$ for all t, with $0 \le \lambda \le 1$, is itself a price sequence that supports the same efficient quantity sequence. The only problem is that the weighted average between two proportional-decline sequences of price vectors need not itself be a proportional-decline sequence; a weighted average of two present value price sequences, each of which has the same own rate of interest across commodities but the two have different such rates, need not itself have the same own rate of interest across commodities. For example, in a two-good, two period model, the price sequence "(1,2) at period 1, (0.5,1) at period 2" has a 100% rate of interest, and "(1,4),(1,4)" has 0%, but their equal-weight average "(1,3), (0.75,2.5)" is not a proportional-decline sequence. It has discount factor 0.75 (own interest rate 0.333...) for commodity 1 and discount factor 0.8333... (own interest rate 0.2) for commodity 2. The defect is not anything fatal to the theory of efficient production plans or their supporting prices, but only to the concept of "the" rate of interest.

All this should have been clear from Malinvaud (1953), and was later made even more abundantly clear by Bliss (1975). But for over a decade, grown men and women got involved in long and arcane exchanges that led nowhere. Allow me to cite a personal memory. In Spring 1977 (when the debate had more or less died down), I was visiting

MIT when one day at the Economics department lunch table the subject came up. Samuelson joked that he could now reveal that all along reswitching had been a neoclassical conspiracy that kept left-wing attackers busy on a pointless technical problem when they might have done some real harm. I only wish that were true.

6. THE OVERLAPPING GENERATIONS MODEL

This single paper (Samuelson 1958 [1966]) deserves a section of its own, because it overlaps several fields -- general equilibrium theory, monetary theory, public finance, macroeconomics -- and it made possible major advances in all these fields. [x]

The proximate purpose of the paper was to examine equilibrium and efficiency in an economy that lasts forever, but where each individual has a finite life, and there is no single market where all can trade simultaneously. The paper also exhibits key properties of the MIT style of small purpose-built models that I mentioned above in Section 4. In reality, individual lifetimes are long, varied and uncertain, but those complications are not essential for the issue being examined. Therefore Samuelson assumes that a generation or cohort of individuals is born in each period, and each individual lives for three periods, which is the minimum needed for pure exchange trades across generations to be possible with perishable goods. Later models with storable goods or durable capital have taken this one step further and assumed two-period lives.

Suppose each three-period-lived person works to produce and earn one unit of output when young and middle-aged, and retires when old. Let R_t denote the discount factor between periods t and $t+1$. Then a person born at time t chooses consumption quantities C_1, C_2 and C_3 over his lifetime to maximize utility $U(C_1, C_2, C_3)$ subject to the budget constraint $C_1 + R_t\, C_2 + \mathrm{R}_\mathrm{t}\, \mathrm{R}_{\mathrm{t}+1}\, C_3 = 1 + 1 \times R_t$. This yields net saving functions $S_1(R_t, R_{t+1})$, $S_2(R_t, R_{t+1})$ and $S_3(R_t, R_{t+1})$. They are linked by an identity imposed by the budget constraint:

$$S_1(R_t, R_{t+1}) + R_t\, S_2(R_t, R_{t+1}) + R_t\, R_{t+1}\, S_3(R_t, R_{t+1}) = 0.$$

If N_t people are born at time t, the equilibrium condition for this period is

$$N_t\, S_1(R_t, R_{t+1}) + N_{t-1}\, S_2(R_{t-1}, R_t) + N_{t-2}\, S_3(R_{t-2}, R_{t-1}) = 0.$$

Consider a geometrically growing population: $N_t = (1+g)^t$, and try a solution with a constant interest rate: $R_t = (1+i)^{-t}$. Comparing the equilbrium condition and the budget identity, we see that $i = g$ is such an equilibrium. Samuelson offers this as a rigorous formulation of Böhm-Bawerk's biological theory of interest. More importantly, he shows that, among all possible "steady state" configurations where per capita consumption quantities at the various periods of life are kept constant across generations, this equilibrium maximizes the utility of the representative person -- an early appearance in the economics literature of the "golden rule of growth."

Unfortunately things are not so simple. Abbreviate $B = 1+g$ and try an equilibrium with a constant R. The budget identity and the equilibrium condition,

$$S_1(R,R) + R\,S_2(R,R) + R^2\;S_3(R,R) = 0 \quad \text{and} \quad B^2\;S_1(R,R) + B\,S_2(R,R) + S_3(R,R) = 0,$$

combine to yield $[B\;S_1(R,R) - R\,S_3(R,R)](1-BR) = 0$. There are two possibilities: $R = 1/B$, which is the golden rule, or $B\;S_1(R,R) - R\,S_3(R,R) = 0$. With some examples in Samuelson (1958 [1966]), and more explicitly in an exchange with Meckling (1960) (see Samuelson 1960 [1966]), he proves that the golden rule state can never be sustained or approached. Even the other state has instability: paths that start slightly away from it will diverge further from it. Worse, there can be multiple equilibria; even for a geometrically growing population, there can be equilibria with different constant rates of interest as well as ones with time-varying rates of interest.

Some general, albeit negative, properties can be inferred. Equilibria typically are not Pareto optimal. One may be able to increase the utility of the generation born at t by making a transfer from the one born at $t+1$, compensate the latter by making a transfer from the one born at $t+2$, ... ad infinitum. Optimality can be achieved by introducing a storable asset even if it has no intrinsic value, such as paper money. [xi]

Second, determination of equilibrium is essentially forward-looking. Start the economy at time t, so at this time there are only the young, and at time $t+1$ only the young and the middle-aged are alive. Then equilibrium conditions proceed

$$S_1(R_1,R_2) = 0, B_2\,S_1(R_2,R_3) + B_1\,S_2(R_1,R_2) = 0, B_3\,S_1(R_3,R_4) + B_2\,S_2(R_2,R_3) + B_1\,S_3(R_1,R_3) = 0,$$

and so on. If we knew R_1, we could successively find R_t for $t = 2,3,\cdots$ But we don't, so we have one degree of freedom. The solution must be completed using a terminal or transversality condition: "[T]oday's interest rate is determined simultaneously with -- and not prior to -- *all* subsequent interest rates." (Samuelson 1960 [1966], 246; emphasis in the original)

Ironically in view of Samuelson's preference for concrete, application-specific models of general equilibrium, his work on this one remained at a relatively abstract level, while the model really took off in the literature only when Diamond (1965) modified it to allow fixed capital and growth, thus paving the way for applications to issues like social security, and when Azariadis (1981), Cass and Shell (1983) and others built versions of it that enabled it to become a staple of macroeconomic models with rational expectations. [xii]

Samuelson participated occasionally in the discussions on social security that used the overlapping generations structure. He pointed out the "beauty" of the "actuarially unsound" unfunded schemes, relying on the growth of population and productivity: "A growing nation is the greatest Ponzi scheme ever contrived" (1967).[xiii] But he was very aware that the "primrose path" of such of unfunded programs becomes "strewn with thorns" if population growth slows or declines (1985, 442).

7. PUBLIC ECONOMICS

In a widely circulated 1951 "memorandum for the U.S. treasury," published 35 years later,[xiv] Samuelson (1986 b [2011]) revisited and extended Ramsey's (1927) theory of optimal commodity taxation, derived the rule of equiproportionate reduction in the compensated demands for all goods, and set the stage for the large body of research that continued through the landmark work of Diamond and Mirrlees (1971) and beyond.

But his most important contribution in this field is undoubtedly to the theory of public goods. To quote Musgrave (1983, 141), this theory "may be dated from June 1954, when Samuelson's 'Pure Theory of Public Expenditure' appeared. Never have three pages had so great an impact on the theory of public finance." Actually Musgrave had a related prior paper based on Lindahl's work -- Samuelson cites these -- but the clarity and simplicity of Samuelson's model and statement of results earned him justified credit. He established (Samuelson 1954 b and 1955 [1966]) the Pareto optimality condition for public goods, summation of individuals' marginal rates of substitution, and its geometric equivalent, vertical summation of demand curves. He makes the connection, here and later more explicitly in Samuelson (1969 a [1972]), with the individualized pseudo-tax prices of Lindahl, as well as to Ramsey pricing related to the theory of optimal commodity taxation mentioned above.

What enabled Samuelson to achieve this clarity? It was his "small tailored model" style of focusing on the issue at hand -- here that of Pareto optimality -- and ignoring other aspects -- here the government budget and financing the public good, and perhaps more importantly, truthful revelation of individual preferences for the public good. He was well aware of the issues, and had the right ideas about them outside of the model. For example, in an exchange with James Buchanan (Samuelson 1968 [1972], 522-523), he says: "Each man knows this [marginal utility] function for himself. But only God knows them for all men. ... If the unknown marginal-utility functions appertained to private goods, this lack of knowledge would not matter. ... It would be in each man's interest to act along his MU function as an observable demand function." [xv] He goes on to discuss informally various ways and costs of getting such information for public goods, but rigorous work on demand-revealing mechanisms was then only in its infancy.

8. FINANCIAL ECONOMICS

Arrow (2006, xii) said of Samuelson's research in this field: "The observation that there was little predictability of securities prices worked powerfully on Paul's analytically-shaped imagination. It led through a series of steps to a theory of warrant prices, which, in turn, inspired Fischer Black, Robert Merton, and Myron Scholes to the complete rethinking (or perhaps, first clear thinking) of the evolution of prices of securities and their derivatives and, with that, a move to practical application unprecedented in the history of economics." Indeed, if impact of economics research is measured by the monetary value of the change it brought about in the economy, it is hard to think of anything bigger. [xvi] As befits work of such importance, we have not one but two definitive accounts and assessments of this work from the person best qualified to give them, Samuelson's star student and coauthor in this field, Robert Merton (1983 and 2006). Therefore I will merely mention a few highlights.

In Samuelson (1965 a, 1971 d [1972]) and some later papers, he examined the efficient or shadow prices of assets whose payoffs come at some time in the future, and their possible relationship to prices in financial markets for such assets. The key idea was that changes over time in such prices could only occur in response to the arrival of new information (or new shocks) -- any previously available information would be already reflected in the price. This idea burgeoned into the "efficient markets" and "random walk" hypotheses.

Two papers (Samuelson 1967 a and 1971 b [1972]) concern the seemingly mundane topic of portfolio selection, but they are important definitive statements; the first establishes the desirability of diversification under very general conditions, the second brings dynamic optimization methods to the subject. And Samuelson (1967 b [1972]) is a prescient model of portfolio selection when the assets have fat-tailed distributions, a topic whose importance the profession realized much later.

His papers on warrant pricing (Samuelson 1965 b [1972], Samuelson and Merton 1969 [1972]) helped launch the whole theory and practice of derivative securities and their pricing. The appendix contributed by his mathematician colleague Henry P. McKean Jr. gave us the technique -- the high-order contact or smooth pasting condition -- that is key to solving these problems. But the idea came to fruition only after Black and Scholes and Merton developed the key idea of dynamic hedging -- that a riskless asset can be constructed by holding the option or warrant long and a suitable quantity of the underlying asset short, and changing the "suitable quantity" in just the right way continuously in time. The no-arbitrage condition this implies enables us to replace the subjective rate of time discount in the partial differential equation that governs the dynamics of the derivative price by the objective riskless rate of interest in the market; that is a crucial step in bringing the procedure into the realm of practicality. Samuelson often expressed some chagrin at his own failure to see this, and called it a near-miss (Merton 2006, 285).

Financial economics, even more than other fields like international trade, is susceptible to misunderstanding, outright error, and potentially very costly misapplication of its ideas. Probably the most egregious of these is the mistaken inference from a vague "law of large numbers" to argue that stocks are essentially riskless in the long run, and that therefore lifelong investing, for example for pensions, should concentrate on stocks most of the time. Samuelson (1969 b [1972]) already produced examples in which such behavior is not optimal, but he went on to write several further papers, both formally (for example 1997 a [2011]) and in more practical terms (for example 1994 [2011]). [xvii] At an even more basic level, he wrote several pieces for the general public (for example 1981 [2011]) giving wise basic advice on investing -- diversify using mutual funds, keep fees low, home-buying is not always the best strategy, and so on.

9. MACROECONOMICS, MONETARY AND FISCAL POLICY

In *Foundations,* Samuelson (1947, 276-283) gave us a clear and compact exposition of the IS-LM model that had been developed in the late 1930s by Meade, Hicks, and Lange. But his original early contribution to macroeconomics is surely the thorough development of the multiplier-accelerator model. In two papers (Samuelson 1939 c and 1939 d [1966]), he converted Hansen's numerical examples into an algebraic second-order difference equation and characterized its behavior for different sets of parameter values; the second of these papers introduced the Keyensian cross diagram. In subsequent papers he developed the model further, and even made some contributions to the mathematical theory of stability of difference equations, for example Samuelson (1941 and 1942 [1966]). And in *Foundations* (Samuelson 1947, Chs. X, XI and Mathematical Appendix B) he brought together this material into a masterly exposition that still repays study. Samuelson (1948 c) has a synthesis and survey of the Keyensian cross model that is still worth reading. His textbook *Economics* (Samuelson 1948 a) popularized the model, especially concepts such as the paradox of thrift.

Of Samuelson's contributions to macroeconomics, his paper with Solow on inflation and the Phillips curve (Samuelson and Solow 1960) probably has had the most impact and generated the most controversy. The article was widely interpreted as lending support to the ideas of a tradeoff between unemployment and inflation, and of choosing macroeconomic policies to achieve a desired or socially optimal point along this curve.

These ideas were criticized by Friedman (1968, 7-11), and later (in even more dramatic terms) by the rational expectations school. (Friedman mentions the Phillips curve and argues against its stability and the feasibility of making policy based on it, but does not cite the Samuelson-Solow article.)

Re-reading the article, I was struck by the cautious tone of the authors, and their awareness of many criticisms later levied against the Phillips curve and others besides. For most of the article their focus is on the statistical difficulties of distinguishing hypotheses about, and identifying causes of, inflation (demand-pull versus cost-push etc.). When in the last two pages they turn to policy matters, specifically to using demand deflating policies to reduce inflation, they clearly state that they are offering "simply our best guesses" and go on: "[W]e must give another caution. All of our discussion has been phrased in short-run terms. ... What we do in a policy way during the next few years might cause [the curve] to shift". They list several such possibilities: (1) "[I]t might be that low-pressure demand would so act upon wage and other expectations to shift the curve downward in the longer run." [xviii] (2) "But the opposite is also conceivable. A low-pressure economy might build up within itself over the years larger and larger amounts of structural unemployment," shifting the trade-off adversely and "more and more unemployment being needed just to keep prices stable." (3) In the long run a low-pressure economy could succeed in improving the efficiency of our productive factors." (4) But a low-demand, high-unemployment economy might lead to "class warfare and social conflict and depressed levels of research and technical progress." (5) Institutional reforms might arise to cope with the changed conditions of low pressure of demand and high unemployment. (Samuelson and Solow 1960, 1352-1353). In the face of these cautions, it would take a brave reader to read an unqualified message of a stable unemployment-inflation tradeoff in this article, but many readers seem to have done so. Perhaps their simple Figure 2 was too beguiling. One picture is worth a thousand words, and in this case their picture seems to have drowned out five hundred words of caution.

10. HISTORY OF ECONOMICS

Samuelson had deep knowledge of, and an abiding interest in, the history and evolution of the subject of economics. To quote Tobin (1983, 191), "His feeling for economics as an evolving science with history and tradition is rare, all too rare, among modern economists."

At a frivolous level, we see this in his use of "freight-train" names for theories and models to ensure proper assignment of priority and credit: the Wong-Viner envelope theorem, the Heckscher-Ohlin model, and so on. In the process of discussing and naming these ideas, he made new contributions of such importance that later researchers attached his name at the end of the train. So we now have the Wong-Viner-Samuelson envelope theorem, the Heckscher-Ohlin-Samuelson model, and so on. If Samuelson had been more of a game theorist, I would have suspected that freight-train naming was a deliberate strategy, derived by backward induction anticipating these future developments!

More seriously, he spent much time and effort understanding the writings of previous giants, trying to clarify and exposit them for the rest of us, and correcting them where that was called for. Perhaps he took exegesis and Dogmengeschichte beyond all reasonable limits; he wrote no fewer than 20 articles on Marx (several of them on arcana of the transformation problem), 7 on Ricardo, and 11 on Sraffa. It is a matter of taste, but I would have preferred him to devote that time to producing more original Samuelsonian gems instead.

von Weizsäcker (1983) argues that Samuelson's efforts had value. "The basic mistake made by Ricardo and Marx was not to distinguish clearly between exogenous data and endogenous variables. The inconsistencies shown by Samuelson in Marx and Ricardo can, it seems to me, all be explained by this mistake." It should not have taken so many articles to demolish such an elementary error. In any case, this is probably an arena in which rational discourse is not possible; people on each side will go on believing what they want to believe.

11. THE TEXTBOOK

Soon after its publication, *Economics* (Samuelson 1948 a) was widely hailed as a breath of fresh air in a world of turgid old economics textbooks. Its lively style, clear but rigorous explanation of concepts and analytical tools, and presentation of facts in well-designed figures and tables, still stand as models for would-be competitors. But today it is remembered above all for introducing Keynes' ideas to undergraduates and thence to the wider public, thus popularizing the Keynesian revolution that was already taking hold among the younger economists of the time.

Keynesian ideas imply a greater role for government in the economy than is believed proper by laissez-faire ideologists. However, the book also devotes considerable space and effort to explaining how markets function and how they solve the central problems of economic organization: what to produce, how to produce, and for whom to produce (Samuelson 1948 a, 35): "a competitive system of markets and prices -- whatever else it may be, however imperfectly it may function -- is not a system of chaos and anarchy. There is in it certain order and orderliness. It works."

With its recognition of the value of both governments and markets, the book came under heavy criticism from both extremes of right and left. [xix] But most non-extreme people would agree that Samuelson had a good, balanced judgment about the relative merits of markets and governments. Neither is perfect, and each needs checks and balances from the other. Most importantly, he argued that markets cannot ensure that the economy will operate at or near its full-employment potential; that is where government policy to sustain the proper level of aggregate demand is important. However, he also argued that when this is done, markets can allocate resources to meet the various components of this aggregate demand reasonably efficiently, allowing for some distortions caused by monopolies and oligopolies, externalities and public goods.

This position is now labeled the "neoclassical synthesis," and Samuelson generally gets credit for the idea. The phrase seems to appear for the first time in the third edition of *Economics* (see Blanchard 2008 and Patinkin 1983, 163), but we find a germ of the idea in the first edition: "When there is substantially full employment, certain important economic principles are valid." (Samuelson 1948 a, 9) and "The important hard kernel of truth in the older economics of full employment can ... be separated from the chaff of misleading applications. ... [I]f modern economics does its task well so that widespread unemployment is substantially banished from democratic societies, then its importance will wither away and traditional economics (whose concern is the *wise* allocation of fully employed resources) will really come into its own" (Samuelson 1948 a, 10; emphasis in the original). Tobin (1983, 197-200) discusses the impact and policy implications of this idea, as well the criticisms from left and right that it attracted.

Can we trace the neoclassical synthesis any farther back? Yes; in fact to the *General Theory* itself (Keynes 1936 [1978]). (Not that Samuelson was claiming anything new or original in his statement.) Here is what Keynes would call a "portmanteau quotation" from the last four or five pages of his book:

"But if our central controls succeed in establishing an aggregate volume of output corresponding to full employment as nearly as is practicable, the classical theory comes into its own again from this point onwards. If we suppose the volume of output to be given, i.e. to be determined by forces outside the classical theme of thought, then there is no objection to be raised against the classical analysis of the manner in which private self-interest will determine what in particular is produced, in what proportions the factors of production will be combined to produce it, and how the value of the final product will be distributed between them. ... Thus, apart from the necessity of central controls to bring about an adjustment between the propensity to save and the inducement to invest, there is no reason to socialise economic life than there was before.

"The central controls necessary to ensure full employment will, of course, involve a large extension of the traditional functions of government. Furthermore, modern classical theory has itself called attention to various conditions in which the free play of economic forces may need to be curbed or guided. But there will still remain a wide field for the exercise of private initiative and responsibility. Within this field the traditional advantages of individualism will still hold good. ... They are partly advantages of efficiency -- the advantages of decentralization and the play of self-interest. ... It is also the best safeguard for the variety of life ... the loss of which is the greatest of all the losses of the homogeneous or totalitarian state.

"The authoritarian state systems of today seem to solve the problem of unemployment at the expense of efficiency and of freedom. ... [I]t may be possible by a right analysis of the problem to cure the disease while preserving efficiency and freedom."

The intrinsic value of freedom also finds an echo in Samuelson's thinking. This aspect, perhaps more importantly than economic efficiency, may have been decisive in his preference for using markets whenever they function reasonably well: "What I learned from the McCarthy incident was the perils of a one-employer society. When you are blackballed from government employment, there is great safety in the existence of thousands of anonymous employers out there in the market. ... To me this became a newly perceived argument, not so much for laissez faire capitalism as, for the *mixed* economy." (Samuelson 1983 b [1986], 790; emphasis in the original.)

12. STUDENTS

Surprisingly few PhD students at MIT had Samuelson as their sole or primary advisor. But they did include two future Nobel laureates, Klein and Merton. He co-supervised many other students with Solow, Modigliani and other colleagues; the list includes Joseph Stiglitz and Stanley Fischer. And he influenced almost all economics students at MIT (and many elsewhere) through his teaching, comments on papers, and questions in workshops. Perhaps his style was not suitable for guiding PhD theses; when reading a student's paper he thought of a dozen ideas tangential to it or arising from it, instead of focusing on how the paper could be turned into a dissertation chapter.

For decades he was the guiding light of the MIT economics department, and surely deserves a large part of the credit for taking it from almost nothing to top levels in world rankings. This was not any part of a grand design when he moved there from Harvard. The initial energy and entrepreneurship came from Rupert Maclaurin, an economics professor and son of former MIT President Richard Maclaurin. He raised money to launch a separate department within the social sciences, with two branches: "industrial relations" and "industrial economics," clearly intended for teaching engineers rather than for research. Statistics professor Harold Freeman was instrumental in recruiting Samuelson (and Solow, too!). Good graduate students began to trickle in attracted by his name, and then more came as the first ones spread good word about the department. Samuelson devoted much effort to his teaching. His colleagues remember him saying: "We take good care of them and that gets around" and "We can't compete with Harvard and Yale financially, so we have to out-think them." He and Solow established the ethos of the department that the graduate program took central place. No one bought off teaching time with outside funds. No other faculty member could act like a prima donna when Samuelson was not acting like one.

I can speak about his classroom teaching from personal experience. Even now the memories make me smile. He always lectured at 9 a.m. We students were still fighting sleep, but he would walk in fresh, carrying freshly prepared handouts. He wrote them out on blue spirit-duplicator masters in the hour before class -- some said he wrote them in his car when it was stopped at red lights -- and ran off copies himself in the department's machine room. They were still damp, and smelled of duplicator fluid so strongly that one could get high on them. But our real mental high came from the material, and from Samuelson's unique delivery.

His lectures were not the best way to learn the basics. But if we came prepared with our advance reading, he showed us subtleties and nuances of the subject in a way no one else could. We began to understand modern economic theory in the context of its origins from its leading pioneer. Most importantly, we acquired a way of thinking about economics that would last a lifetime. We learned methods and skills for research that the cut-and-dried world of the textbooks left out.

This was interspersed with memorable and funny anecdotes about famous economists. Smith, Ricardo, Marshall, Edgeworth, Keynes, Schumpeter, Irving Fisher, and above all Frank Ramsey, came alive for us in a way that taught us to respect the history of the subject and to appreciate the height of the shoulders of these giants, while at the same time making us smile at their human foibles. Irving Fisher's touching faith in the permanence of stock market valuations just before the 1929 crash was matched by Joan Robinson's equally touching faith in everything Chairman Mao told her about the Chinese economy. Speaking of Joan Robinson, who can forget Samuelson's caricature of her visit to the United States: "She was taken in a sealed train from coast to coast -- from Paul Baran to Paul Sweezy." [xx]

His favorite by far was Frank Ramsey. He told us how, on his first day as an undergraduate at Cambridge, Ramsey met with his philosophy tutor Ogden, and started to discuss some of his ideas about essence and being. After listening, Ogden said, "These notions are rather like those of Kant." "Kant? Who is he?" was Ramsey's reply. "Emanuel Kant was the author of this book I'll lend you, *The Critique of Pure Reason*." "But it's in German, sir, and I don't know any German." "That's all right, I'll lend you this dictionary." A couple of weeks later Ramsey came back to Ogden saying "Kant has it almost right, but …" Samuelson clearly reveled in telling this story, recognizing in Ramsey a kindred spirit in precocity and genius.

The single most important thing I learned from Samuelson, and have tried to use in my research, is a sense of unity of the subject – of economics, and of mathematical methods used in economic analysis. From his own work and in his teaching, I learned that all the "fields" into which economics is conventionally divided are intricately linked pieces of one big puzzle, with a common framework of concepts and techniques -- choice, equilibrium, and dynamics. The same goes for intrinsic links between economics and mathematics. Samuelson conveyed this via another anecdote, this one about J. Willard Gibbs, the famous mathematical physicist. Gibbs was renowned for never saying a word at Yale faculty meetings. But once when a proposal to let students choose either a foreign language or mathematics was being discussed, Gibbs spoke up: "Mathematics is a language." Samuelson made this the title-page quote for *Foundations,* and later (1952 b [1966]) improved it to "Mathematics *is* language" (emphasis in the original). This language is particularly well suited to formulating and understanding the multidimensional complexities of economic interactions, where many entities are interconnected by bidirectional links of cause and effect, and randomness has a big role.

13. INTERACTION WITH THE PROFESSION

Samuelson took his role as an active member of the profession with utmost seriousness. Anyone who wrote to him with a good question got his serious attention and a thoughtful answer. He read the research literature voraciously, and wrote to many of the authors, offering praise, questions, or counterarguments as appropriate. He abhorred error and sloppy thinking, and spent much time trying to correct errors wherever he saw them, but he was extremely generous with his praise, too. His collected works include almost 50 Festschrift contributions, and appreciations, remembrances, memorials and obituaries of over 60 economists; I have to admit that I had never even heard of some of these people.

14. POPULAR WRITING AND POLICY INVOLVEMENT

Samuelson attached great importance to conveying economists' ideas to the world outside the profession. Correcting popular errors and misconceptions about economics and economic policy was vital, especially in areas like inflation and unemployment, international trade, and personal investing, where errors can have huge consequences for individuals and nations. He was as prolific in writing for the public as in his research: his collected works contain over 75 essays in magazines and books aimed at the intelligentsia and policymakers, op-ed articles, texts of speeches and congressional testimonies, and so on. He was a regular columnist for *Newsweek* from September 1966 to May 1981. He wrote almost 250 of these columns, for many years in a three-week rotation with Milton Friedman and Henry Wallich and much sparring with Friedman. Only a few of these columns are reprinted in his collected works, but most can be found in Samuelson (1973 and 1983 c).

15. CONCLUDING REMARK

Re-reading Samuelson's work for the purpose of writing this article redoubled my sense of amazement and awe. How could one man in one lifetime achieve so much (and also devote plenty of quality time to family (six children including a set of triplets!), friends, colleagues, students, and his beloved institutions -- the economics profession, MIT, and its economics department)? For possible answers I can do no better than to quote Merton (2006, 293): "First, there is his seemingly infinite capacity for problem finding and his supersaturated knowledge of just about every special sphere of economics. Second, there is his speed of problem solving together with the ability to put the solution quickly to paper with great skill, great verve, and lack of hesitation. Third, strong opinions and decisive language are characteristics of Samuelson writings, and yet it is his willingness to change his views and admit errors that makes his steadfastness on some issues so credible. Finally, although often masked by the apparent ease with which he produces, there is his diligence. Paul has always worked hard. ... Even at play, he is at work."

Frank Hahn (1983) concluded his essay in the first Samuelson Festschrift thus: "If I have expressed reservations about an aspect of Samuelson's methodology or here and there allowed a critical note of this or that construction, it is at least partly to avoid the pitfalls of hagiography into which any student of Samuelson's work is only too apt to fall." In this article I have fallen far into that pit, but that is perhaps appropriate for someone who was not only a student of Samuelson's work, but also a student in the direct personal sense, and during some visiting semesters, a very junior awestruck colleague.

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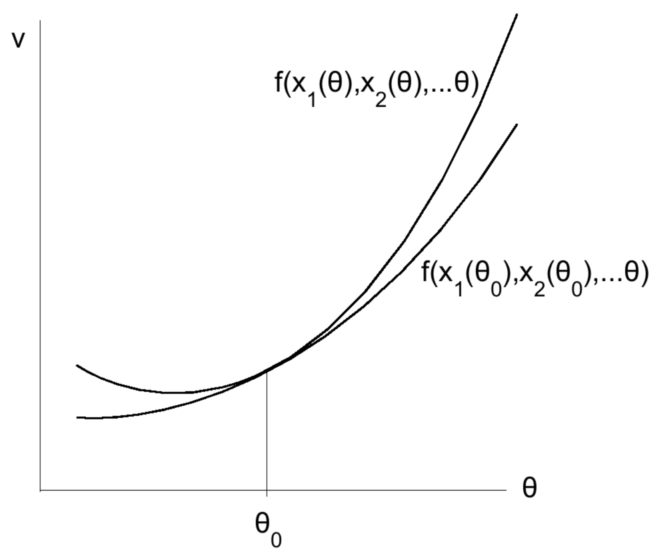
Figure 1: Envelope Theorem and Le Chatelier Principle



FOOTNOTES

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[i] Bartlett's Familiar Quotations Online, http://www.bartleby.com/100/230.185.html

[ii] http://nobelprize.org/nobel_prizes/economics/laureates/1970/

[iii] I will cite Samuelson's work by the original publication, and the reprinting in the volume of his collected works where applicable. In those cases, page references are to the latter.

[iv] My account of Samuelson's biographical and personal aspects, in this section and elsewhere in this article, is based mainly on Brown and Solow (1983) and Samuelson (1983 a), but also on various published obituaries, and on conversations with Samuelson's close colleagues, especially Robert Solow, and on my own recollections. More detailed biographical information can be found in Weinstein (2009), on the web site of the Nobel Foundation:
http://nobelprize.org/nobel_prizes/economics/laureates/1970/samuelson-bio.html ,
and on Wikipedia, http://en.wikipedia.org/wiki/Paul_Samuelson

[v] His life span of 94 years, 7 months and 26 days was therefore over four months longer than that of his friend and opponent in many arguments, Milton Friedman (July 31, 1912 - November 16, 2006).

[vi] A physicist who played a major role in the development of the hydrogen bomb.

[vii] It also underlies *The Economist* magazine's "big mac index" of departures from PPP. For an explanation, see http://en.wikipedia.org/wiki/Big_Mac_Index . The latest update is at http://www.economist.com/blogs/dailychart/2011/07/big-mac-index .

[viii] The first of these articles is in fact his first appearance in a major economics journal. He was then three months shy of 22! Aready it contains (1937 a, 216-217) an "ordinalist manifesto" that has guided much of his thinking and writing on consumer theory ever after. Even more remarkably, in his concluding comments (1937 a, 217), Samuelson suggests the possibility of time inconsistency and commitment strategies for dealing with it -- issues that were taken up only much later by the profession.

[ix] I shall refrain from making any *Matrix* jokes, but leave that as a suggestion for others.

[x] Perhaps this versatility explains why it fell through the cracks in the first Festschrift (eds. Brown and Solow 1983), which was organized along conventional field lines; see Brown and Solow (1983, ix). That error of omission was remedied in the second Festschrift (Solow 2006). Geanakoplos (2008) explains the model and reviews its subsequent impact.

[xi] Further development with some simple examples is in Shell (1971).

[xii] Perhaps to Samuelson's chagrin.

[xiii] This remark enjoyed a revival in 2011, when Rick Perry, then a candidate in the Republican presidential primaries, called social security a Ponzi Scheme but in a negative sense. Many bloggers pointed out Samuelson's different interpretation, for example Alex Tabarrok in Marginal Revolution, http://marginalrevolution.com/marginalrevolution/2011/09/is-social-security-a-ponzi-scheme.html , accessed November 24, 2011.

[xiv] I still treasure my spirit-duplicator copy.

[xv] This is rigorously valid only in the limiting case of a large economy where each individual has negligible influence on aggregate demand, but that is a pedantic qualification here.

[xvi] The Keynesian revolution is the only potential competitor. In natural sciences and engineering we have the transistor, the laser, and DNA.

[xvii] Merton's detailed account of this (2006, 286-291) is highly worth reading.

[xviii] This is the point emphasized by Friedman.

[xix] Nasar (2011, Ch. XVI) describes some of these developments. One left wing critic went so far as to publish *Anti-Samuelson* (Linder 1977). My reaction to that book was the opposite of what the author intended: "This is the best evidence of Samuelson's success! If I had a lot of money, I would secretly pay someone to publish an Anti-Dixit."

[xx] Both were very left-wing economists. For young readers ignorant of history, I should explain that the allusion is to Lenin's journey in a sealed train from Zürich to St. Petersburg in April 1917 to take over leadership of the Russian revolution.