

The Simplest Derivation of $E = mc^2$

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Introduction

The mass-energy equation, $E = mc^2$, is one of the fundamental principles in physics, revealing that mass and energy are equivalent. However, after more than 100 years, the theory of relativity still isn't widely accepted as common knowledge. This issue may stem from our inability to present the information in a more comprehensible manner to the public. Simplified derivations of the mass-energy equation aim to make some concepts easier to understand for high school students, thus making them accessible to the general public. Through these discussions, you'll learn about the limitations of Newtonian physics and gain an appreciation for Einstein's work, which led to a deeper understanding of the universe. Various pieces of evidence and observations make the principle of relativity more concrete and intriguing.

The Simplest Derivation

Maxwell's electromagnetic equations have demonstrated that light travels at a constant speed, denoted as c (approximately 3×10^8 m/s), which has been confirmed through numerous experiments and observations. Additionally, the momentum (p) of electromagnetic waves, or photons, is proportional to their energy level (E):

$$(1) \quad p = \frac{E}{c}$$

or

$$(2) \quad E = pc$$

Many derivations of the above equations are from Einstein's mass-energy equation, which creates a circular dependency. Instead, the recognition and theoretical substantiation of the existence of light momentum and pressure predates the publication of the mass-energy equation and is grounded in empirical observations. Material entities are composed of charged particles, and when an electromagnetic wave incident on such an object, it exerts forces on the charged particles in accordance with the [Lorentz force](#). The energy and momentum of electromagnetic waves are then transmitted through these forces, and this force performs work on the particles, augmenting their energy. Here is a [derivation of equation \(1\)](#) based on a simpler form of Lorentz force. On the other hand, the momentum (p) of an object is defined in classical physics as

$$(3) \quad p = mv$$

where m is the mass of the object, and v is the velocity. In the case of a photon, the definition can be simplified below since a photon travels at the speed of light (c),

$$(4) \quad p = mc$$

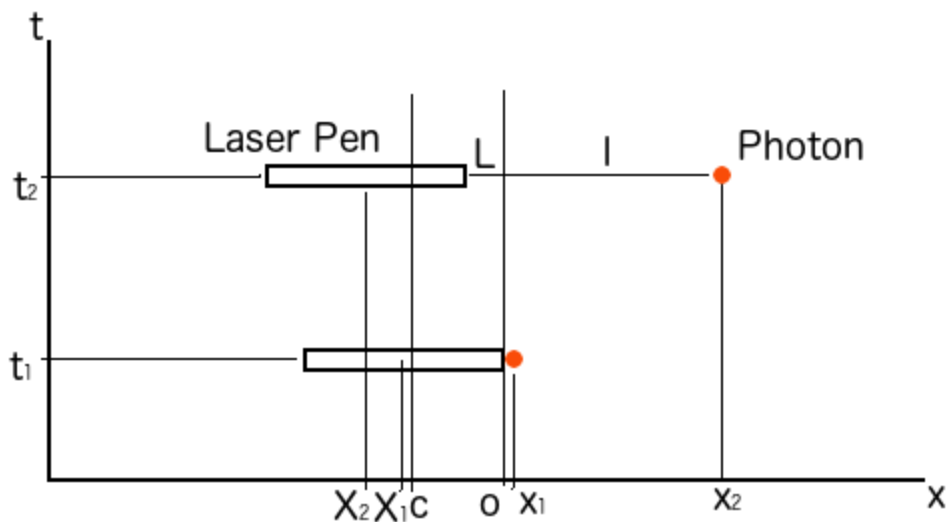
Now, replace p in equation (2) with the momentum given in equation (4), and we found the mass-energy equation:

$$(5) \quad E = mc^2$$

So simple, isn't it? Wait a second, a photon does not have mass. Can we use equation (4) from the previous text in the derivation? The short answer is yes. When people say a photon has no mass, they are referring to the rest/invariant mass, which is typically denoted by m_0 . The mass in the previous equations refers to the relativistic mass. In the case of a photon, there is no rest mass, but it still possesses relativistic mass. For a stationary object, its relativistic mass has the same value as its rest mass. We will demonstrate later that the momentum definition in (3) is generally valid in both Newtonian and relativistic physics when considering m as a relativistic mass. If you're still not convinced by this brief proof, let's proceed to prove equation (4) for you next.

A Proof of $p = mc$ for Photon

Consider a thought experiment with a laser pointer. It is at rest in an inertial frame, as depicted in the figure below, with the x -axis for distance in the horizontal direction and time in the vertical direction. At time t_1 , the laser emits a photon to the right in the horizontal direction. Even though photons have no rest mass, light pressure has been proven and observed in experiments, indicating that there is momentum, and therefore, relativistic mass in photons. As a result, the pointer and the photon move apart as shown at time t_2 . The pointer moves L meters to the left, while the photon moves l meters to the right, away from their original position, o .



The mass center of both pointer and photon was initially at location C which can be calculated like this:

$$(6) \quad C = \frac{MX_1 + mx_1}{M+m}$$

where M and m represent the masses of the pointer and photon, respectively, and X_1 and x_1 indicate the mass centers for the pen and photon. Similarly, the new mass center for both pen and photon at time t_2 can be calculated as

$$(7) \quad C = \frac{MX_2 + mx_2}{M+m} = \frac{M(X_1+L) + m(x_1+l)}{M+m} = \frac{MX_1 - ML + mx_1 + ml}{M+m}$$

where X_2 and x_2 represent the mass centers for the pen and photon at time t_2 . Because there is no external force, both mass centers should also be at the same location. Therefore, we have the equation

$$(8) \quad \frac{MX_1 + mx_1}{M+m} = \frac{MX_1 - ML + mx_1 + ml}{M+m}$$

which is simplified to

$$(9) \quad ML = ml$$

By dividing the time duration ($t_2 - t_1$) from both sides of the equation, it becomes

$$(10) \quad \frac{ML}{t_2 - t_1} = \frac{ml}{t_2 - t_1}$$

Recall that the speed of the laser pointer, V , equals $L/(t_2 - t_1)$, and the speed of the photon, v , equals $l/(t_2 - t_1)$. The equation is then reduced to:

$$(11) \quad MV = mv$$

Equation (11), which we just derived, represents the conservation principle of momentum. Keep in mind that l is the distance the photon traveled during the time interval ($t_2 - t_1$), so the speed v mentioned above is actually the speed of light, c . Therefore, the equation can be rewritten as

$$(12) \quad MV = mc$$

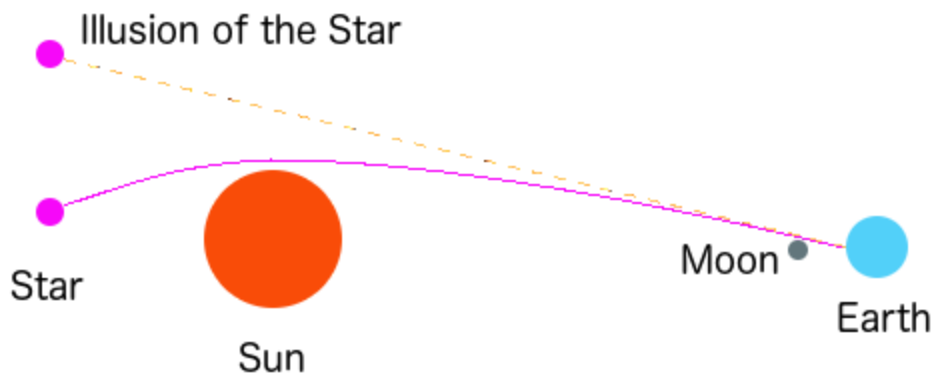
In this context, all the momenta discussed, including both the photon's momentum (p_{photon}) and the pointer's momentum (p_{pointer}), have equal values. By combining these momenta with the definition (3) for momentum, we can establish the following relationships

$$(13) \quad P_{\text{photon}} = P_{\text{pointer}} = MV = mc$$

which proves equation (4), $p = mc$, confirming that definition (3) can be extended from Newtonian to relativistic physics. Therefore, the derivation in the previous section is valid. Next, let us show an observation that demonstrates the effect of the relativistic mass of photons.

An Effect of Relativistic Mass

It is common sense that light travels in a straight line. Can you imagine that light might be bent? Yes, this has been predicted in Einstein's theory of general relativity and was observed in a subsequent experiment. Astronomers mapped the locations of stars with respect to the Sun and Earth in the sky. In one instance, a particular star was situated on the opposite side of the Sun, as shown in the following figure. If the light from the star were to travel in a straight line, it would be blocked by the Sun, making it impossible for an observer on Earth to see the star's light. However, during a solar eclipse, the star was observed, creating an illusion as indicated by the dotted line.



As we all know, there is energy in light, which is given by

$$(14) \quad E = hf$$

where h represents Planck's constant and f is the frequency of light. With the mass-energy equation (5), we can calculate the relativistic mass

$$(15) \quad m = \frac{hf}{c^2}$$

for the light with a frequency f . Is the curved light due to the attraction of the Sun? Can we apply relativistic mass to Newton's law of universal gravity? This is the part where the law of universal gravity needs to be adjusted.

Newton's law of gravitation is highly useful in predicting the orbits of celestial bodies. However, the predicted orbit for Mercury always deviated slightly from measurements. Physicists were puzzled for a long time. The mystery was solved by Einstein in his general relativity theory. Starting from a postulation that the inertial mass and gravitational mass are equivalent, Einstein was able to derive the theory of general relativity. In the theory, gravity is adjusted with a relativistic term, which accurately predicts Mercury's orbit and the curvature of light in gravitational fields. Thus, the curved light can be considered a result of the Sun's attraction.

In general relativity, Einstein explained these phenomena from a different perspective: gravity curves the space around it, and light travels through the shortest path in this curved space, as do the orbits of celestial bodies. Due to the extreme gravity near black holes, light is so curved that it becomes trapped within the black holes. As a result, we cannot see black holes directly but can observe their effect on nearby objects.

An Intuitive Proof for Ordinary Objects

We just proved the mass-energy equation for photons, which do not have rest mass. Considering the perspective of the laser pointer in the previous derivation, the pointer loses energy E_l , which is the same amount of energy that the photon takes away from the pointer. Thus, the energy loss is

$$(16) \quad E_l = mc^2$$

or, the equivalent of loss in mass,

$$(17) \quad m_l = \frac{E_l}{c^2}$$

This is what Einstein was trying to prove in 1905. However, his logic is difficult for many people to comprehend. Nevertheless, the derivation still has not provided direct proof of the validity of the mass-energy equation for ordinary objects that have rest mass. Next, let's provide an intuitive derivation. Consider a force F exerted on an object. Its momentum will increase. The energy gain dE_g is the work that the force F produces over a distance S

$$(18) \quad dE_g = FdS$$

E_g can be computed by integrating equation (18) over the distance S . This is one way to derive the mass-energy equation in many [textbooks](#), which will not be repeated here. Instead, let's take the derivative of the momentum definition (3) with respect to time,

$$(19) \quad \frac{dp}{dt} = \frac{d(mv)}{dt}$$

In Newtonian physics, the mass m is assumed to be invariant. So, it becomes,

$$(20) \quad \frac{d(mv)}{dt} = \frac{mdv}{dt} = ma = F$$

where a represents the acceleration due to force F that causes the acceleration for mass m . The right part of the equation is Newton's second law, which fails at high speeds because mass is no longer invariant. Let's redefine the second law as the increment of momentum due to the applied force,

$$(21) \quad F = \frac{dp}{dt}$$

This definition works in both Newtonian and relativistic physics when the mass is considered relativistic. With the momentum definition (3), Newton's second law becomes

$$(22) \quad F = \frac{d(mv)}{dt} = \frac{mdv}{dt} + \frac{vdm}{dt}$$

where the first term is the classical portion of Newton's second law, while the second term indicates the mass appreciation due to the applied force. Replace the force F in (18), and the energy gain becomes

$$(23) \quad dE_g = \frac{d(mv)}{dt} dS = \frac{dS}{dt} d(mv) = v d(mv) = v v dm + m v dv$$

or simplified like this

$$(24) \quad dE_g = v^2 dm + \frac{1}{2} m dv^2$$

To find the energy gain, we can take the integral over (24). At low speeds, there is not much change in mass. The first term may be neglected. The energy gain becomes

$$(25) \quad E_g = \frac{1}{2} m v^2 = E_k$$

which is the kinetic energy in Newtonian physics. That is, all the energy gain is contributed to kinetic energy. Equation (24) is generally valid for both Newtonian and relativistic physics for objects at any speed. Because no object can surpass the speed of light, the speed of any object cannot increase beyond the speed of light. Therefore, $dv^2 = 0$ at the speed of light. The second term in (24) becomes zero. At the speed of light, the equation is simplified as

$$(26) \quad dE_g = c^2 dm$$

Taking the integral over (26) for mass from rest m_0 to relativistic m , we find the kinetic energy gain from 0 to E_k ,

$$(27) \quad E_k - 0 = c^2(m - m_0) = mc^2 - m_0c^2$$

or

$$(28) \quad E = mc^2 = E_k + m_0c^2$$

where m_0 is the rest mass, and m_0c^2 represents the rest energy of the object. Equation (28) is just a different way to express the mass-energy equation (5) with a separating term for the rest energy from the kinetic energy as a result of speed gain. In the case of a photon, there is no rest mass and no rest energy.

With equation (24), we can identify that the energy gain of the mass due to the applied force is contributed in two parts: the kinetic energy due to speed acceleration, mostly at low speeds, and the mass increase, primarily at high speeds. Newton's laws are an approximation at low speeds and fail at high speeds. The relativistic physics from Einstein's theories corrects the problems and works universally in modern physics. For a formal derivation of the mass-energy equation with rest mass, please refer to the article [How to Understand Relativity](#).

See Also

- [Unified Theory of Low and High-Temperature Superconductivity \(PDF\)](#)
- [Superfluids Are Not Fluids \(PDF\)](#)
- [Electron Tunnel \(PDF\)](#)
- [LK-99 Limitations and Significances \(PDF\) \(中文\)](#)
- [The Cause of Brownian Motion \(PDF\)](#)
- [The Process Driving Crookes Radiometers \(PDF\)](#)
- [Can Temperature Represent Average Kinetic Energy? \(PDF\)](#)
- [Why Phase Transition Temperature Remains Constant \(PDF\)](#)
- [Is Thermal Expansion Due to Particle Vibrating? \(PDF\)](#)
- [The Nature of Absolute Zero Temperature \(PDF\)](#)
- [Misconceptions in Thermodynamics \(PDF\)](#)
- [Superconductor Origin of Earth's Magnetic Field \(PDF\)](#)
- [Tidal Energy Is Not Renewable \(PDF\)](#)
- [How to Understand Relativity \(PDF\)](#)
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